

# Nonlocally looking equations can make nonlinear quantum dynamics local

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A general method for extending a non-dissipative nonlinear Schrödinger and Liouville-von Neumann 1-particle dynamics to an arbitrary number of particles is described. It is shown at a general level that the dynamics so obtained is completely separable, which is the strongest condition one can impose on dynamics of composite systems. It requires that for all initial states (entangled or not) a subsystem not only cannot be influenced by any action undertaken by an observer in a separated system (strong separability), but additionally that the self-consistency condition  $\text{Tr}_2 \circ \phi_{1+2}^t = \phi_1^t \circ \text{Tr}_2$  is fulfilled. It is shown that a correct extension to  $N$  particles involves integro-differential equations which, in spite of their nonlocal appearance, make the theory fully local. As a consequence a much larger class of nonlinearities satisfying the complete separability condition is allowed than has been assumed so far. In particular all nonlinearities of the form  $F(|\psi(x)|)$  are acceptable. This shows that the locality condition does not single out logarithmic or 1-homogeneous nonlinearities.

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## I. INTRODUCTION

In spite of the linearity of the Schrödinger and Liouville-von Neumann equations, nonlinearly evolving states are encountered in quantum mechanics quite often. Typically this is a result of approximations used in a description of collective phenomena (cf. nonlinear interferences in a Bose-Einstein condensate [1]) but no proof has been given so far that it is not the *linearity* that is a result of some approximation. This obvious observation motivated many authors to either look for nonlinear extensions of quantum mechanics, or to try to find an argument against a nonlinear evolution at a fundamental level.

A privileged role in both cases was played by a separability condition. Apparently the first use of the condition can be found in [2]. The authors required that a nonlinearity must allow two separated, noninteracting and uncorrelated subsystems to evolve independently of each other, and this (plus some additional assumptions) led them to the nonlinear term  $\ln(|\psi(x)|)\psi(x)$ . Similar argument (with different additional assumptions) led Haag and Bannier [3] and Weinberg [4] to the 1-homogeneity condition (i.e.  $\psi(x)$  and  $\lambda\psi(x)$  should satisfy the same equation). An important element of these works was the assumption that the systems are uncorrelated, that is their wave function is a product one. Equations that have such a locality property may be called *weakly separable* [5]. It was shown later by Gisin and others [4,8–10] that a weakly separable equation may still violate causality if an initial state of the composite system is entangled. A slightly different in spirit analysis of separability conditions was given in [11] where it was argued that an extension from 1 to  $N$  particles may lead to new effects that become visible for  $N > N_0$ , where  $N_0$  is a parameter characterizing a given hierarchy of theories.

It was originally conjectured that these properties refute any deterministic (i.e. non-stochastic) nonlinear generalization of quantum mechanics. That this is not the case was shown for Weinberg-type quantum mechanics by Polchinski [12] (for pure states) and Jordan [13] (for density matrices). These results were subsequently generalized to a more general class of theories (Lie-Nambu dynamics) by myself in [14]. Therefore, contrary to a rather common belief, there exists a class of nonlinear generalizations of quantum mechanics that does not lead to the locality problems for general (pure-entangled and general-mixed) states. Such theories can be called *strongly separable*. The strongly separable theories considered so far involve equations which are Hamiltonian, which means there exists a Hamiltonian *function* that generates the dynamics. Actually, the solution of the problem proposed by Polchinski in [12] was based on an appropriate choice of this function. In addition, the Polchinski approach was formulated within a finite-dimensional framework. Still, there exists an interesting class of *non-Hamiltonian* and infinite-dimensional nonlinear equations. For example, it is known that all Doebner-Goldin equations that are Hamiltonian are linearizable [15] and therefore their strong separability may not be very interesting. Quite recently the problem of strong separability of non-Hamiltonian Doebner-Goldin equations was addressed in [16]. Using the typical extension of the dynamics from 1 to  $N$  particles the authors showed that at  $t = 0$  the time derivatives (up to the 3rd order) of a 1-particle probability density in position space do not depend on the potential applied to the other particle if the

equation is Galilean-covariant. This *suggests* that Galilean covariance may lead to strong locality in nonrelativistic domain. The result agrees with computer algebraic tests undertaken by Werner [17] who chose oscillator-type potentials and wave functions satisfying certain Gaussian Ansatz. The approach chosen in those works is essentially a third order perturbation theory applied to diagonal elements of reduced density matrices. A technical detail that did not allow these authors to find a general non-perturbative solution was that they investigated a dynamics of diagonal elements of a non-pure reduced density matrix (typical of entangled states) but the formalism they used was devised for state vectors and, hence, they had no control over the behavior of non-pure states and off-diagonal elements of density matrices [18].

In this paper I will give a general and model independent solution of the problem. I will generalize to non-Hamiltonian systems the technique that proved useful in the context of density matrix equations, namely the triple-bracket formalism [14,21]. I will then show how to extend a 1-particle dynamics to  $N$ -particle systems in a strongly separable way for a large class of both Hamiltonian and non-Hamiltonian Schrödinger non-dissipative equations. As opposed to basically all previous papers dealing with pure state nonlinear dynamics (the exception is [12], but it deals with finite dimensional systems) I will not *guess* the “obvious” form of the extension but will *derive* it. This somewhat long path beginning with pure states, then extension via mixtures, and again back to pure states, will be rewarded because the  $N$ -particle form we will get will not be the one one might expect. Its particularly interesting feature is the fact that the equations are *integro-differential*. This feature is implicitly present also in the Polchinski-Jordan formalism but is hidden behind the finite dimensional convention where the integrals do not explicitly show up. The approach is applicable to *all* non-dissipative Doebner-Goldin equations [6,7], the equations discussed in [2–4,22], as well as a large class of nonlinear Schrödinger equations that were not discussed so far.

I will prove simultaneously a much stronger result. By the very construction the formalism is applicable to those Liouville-von Neumann nonlinear equations that reduce to the corresponding nonlinear Schrödinger dynamics on pure states. All these equations will be shown to be not only *strongly separable* but also *completely separable*. By the latter I mean a strongly separable dynamics which additionally satisfies the self-consistency condition

$$\text{Tr}_2 \circ \phi_{1+2}^t = \phi_1^t \circ \text{Tr}_2, \quad (1)$$

where  $\phi_{1+2}^t$  and  $\phi_1^t$  denote the dynamics of the composite system and the subsystem, respectively, and the partial trace  $\text{Tr}_2$  is a map that reduces the dynamics from the large system to the subsystem. Condition (1), which is independent of strong separability, was recently pointed out as an important ingredient of nonlinear dynamics that may be regarded as completely positive [23]. We will also see that not only the probability densities in position space but also the off-diagonal elements of reduced density matrices are independent of details of interaction in the remote systems.

## II. EXAMPLE: HAAG-BANNIER EQUATION AND ITS ALMOST-LIE-POISSON FORM

The general convention I will use was elaborated in detail in [14,24] but to make this work self-contained let us first explain the general scheme on a concrete example, the Haag-Bannier equation [3]. This equation is of the Doebner-Goldin type, that is contains a 1-homogeneous nonlinear term with derivatives, but is simpler:

$$i\hbar\partial_t\psi(a) = \left(-\frac{\hbar^2}{2m}\Delta_a + V(a)\right)\psi(a) + \vec{A}(a)\frac{\bar{\psi}(a)\vec{\nabla}_a\psi(a) - \psi(a)\vec{\nabla}_a\bar{\psi}(a)}{2i|\psi(a)|^2}\psi(a). \quad (2)$$

Its Liouville-von Neumann counterpart is

$$\begin{aligned} i\hbar\partial_t\rho(a, a') = & \left(-\frac{\hbar^2}{2m}\Delta_a + V(a)\right)\rho(a, a') + \vec{A}(a)\frac{\int dy \delta(a-y)\vec{\nabla}_a[\rho(a, y) - \rho(y, a)]}{2i\rho(a, a)}\rho(a, a') \\ & - \left(-\frac{\hbar^2}{2m}\Delta_{a'} + V(a')\right)\rho(a, a') - \vec{A}(a')\frac{\int dy \delta(a'-y)\vec{\nabla}_{a'}[\rho(a', y) - \rho(y, a')]}{2i\rho(a', a')}\rho(a, a'), \end{aligned} \quad (3)$$

where  $\rho(a, a') = \overline{\rho(a', a)}$ , and  $f(a) = \rho(a, a) \geq 0$  is  $d^3a$ -integrable [25] together with all its natural powers  $f(a)^n$ . For  $\rho(a, a') = \psi(a)\bar{\psi}(a')$  (3) reduces to the Schrödinger-Haag-Bannier dynamics. Using the composite index convention described in [14] we can write the equation in a form which is compact and simplifies general calculations. Denote  $\rho_a = \rho(a, a')$  and

$$H^a(\rho) = H(a', a) = K(a', a) + V(a)\delta(a - a') + \vec{A}(a)\frac{\int dy \delta(a-y)\vec{\nabla}_a[\rho(a, y) - \rho(y, a)]}{2i\rho(a, a)}\delta(a - a'). \quad (4)$$

The kinetic kernel satisfies

$$\int dy K(a, y) \rho(y, a') = -\frac{\hbar^2}{2m} \Delta_a \rho(a, a'), \quad (5)$$

and  $K(a, b) = \overline{K(b, a)}$ . All the integrals are in  $\mathbf{R}^3$  i.e.  $da = d^3a$ , etc., and the “summation convention” is applied at the composite index level (two repeated indices are integrated). There is no conflict of notation here because the composite indices are always in their upper or lower positions whereas the 3-dimensional coordinates  $a$  are arguments of functions or distributions. (Notice that the composite indices correspond always to pairs of primed and unprimed 3-dimensional coordinates.) The indices are raised by the metric  $g^{ab}$  working as follows

$$g^{ab} \rho_b = \int db db' \delta(a - b') \delta(b - a') \rho(b, b') = \rho(a', a) = \rho^a.$$

So if  $\rho_a = \rho(a, a')$  then  $\rho^a = \rho(a', a)$ . To lower an index one uses obvious inverse formulas [14].

The Liouville-von Neumann equation can be written in a triple-bracket-type form [14,26,27]

$$i\hbar \partial_t \rho_f = \int db db' dc dc' \left( \underbrace{\delta(f - b') \delta(b - c') \delta(c - f') - \delta(f - c') \delta(b - f') \delta(c - b')}_{\Omega_{fbc}} \right) \underbrace{H(b', b)}_{H^b} \underbrace{\rho(c', c)}_{\rho^c} \quad (6)$$

$$= \Omega_{fbc} H^b \rho^c = \Omega_{abc} \frac{\delta \rho_f}{\delta \rho_a} H^b \rho^c. \quad (7)$$

Let us note that if

$$H^b = \frac{\delta H}{\delta \rho_b} \quad (8)$$

then (7) describes a Lie-Poisson dynamics of a density matrix [28]. If no Hamiltonian function  $H$  satisfying (8) exists, the dynamics will be called almost-Lie-Poisson. The generic case discussed in this Letter is almost-Lie-Poisson. The Weinberg-type and mean-field dynamics discussed in [13,28] are Lie-Poisson. The discussion presented here applies to a general almost-Lie-Poisson dynamics where the structure constants have the following general form [14]

$$\Omega_{abc} = I_{\alpha\beta'} I_{\beta\gamma'} I_{\gamma\alpha'} - I_{\alpha\gamma'} I_{\beta\alpha'} I_{\gamma\beta'} \quad (9)$$

$$\Omega^{abc} = -\omega^{\alpha\beta'} \omega^{\beta\gamma'} \omega^{\gamma\alpha'} + \omega^{\alpha\gamma'} \omega^{\beta\alpha'} \omega^{\gamma\beta'} \quad (10)$$

where  $\omega^{\alpha\alpha'} = \omega^a$  and  $I_{\alpha\alpha'} = I_a$  are, respectively, the symplectic form and the Poisson tensor corresponding to the pure state equation. Here  $\omega^a = \delta(a - a')$ ,  $I_a = \delta(a - a')$  but (9) and (10) are valid also for other Hilbert spaces and equations (cf. [14,24]).

### III. 2-PARTICLE EXTENSION

Consider now a 2-particle system described by the density matrix which, depending on whether the state is pure or general, is in either of the two forms

$$\rho_a = \rho_{a_1 a_2} = \rho(a_1, a_2, a'_1, a'_2) \quad (11)$$

$$\stackrel{\text{pure}}{=} \Psi(a_1, a_2) \overline{\Psi(a'_1, a'_2)}. \quad (12)$$

The reduced density matrices are

$$\rho_{a_1}^I = \int dy \rho(a_1, y, a'_1, y) = \omega^{a_2} \rho_{a_1 a_2} \quad (13)$$

$$\rho_{a_2}^{II} = \int dy \rho(y, a_2, y, a'_2) = \omega^{a_1} \rho_{a_1 a_2}. \quad (14)$$

The 2-particle Liouville-von Neumann-Haag-Bannier equation is given by (7) but with all indices doubled like in (11), (12) with the 2-particle structure constants given explicitly by

$$\begin{aligned}
\Omega_{abc}^{(2)} &= \delta(a_1 - b'_1)\delta(b_1 - c'_1)\delta(c_1 - a'_1) \\
&\quad \times \delta(a_2 - b'_2)\delta(b_2 - c'_2)\delta(c_2 - a'_2) \\
&\quad - \delta(a_1 - c'_1)\delta(b_1 - a'_1)\delta(c_1 - b'_1) \\
&\quad \times \delta(a_2 - c'_2)\delta(b_2 - a'_2)\delta(c_2 - b'_2).
\end{aligned} \tag{15}$$

The crucial element of the whole construction is the nonlinear extension of the 1-particle Hamiltonian operator kernel. We *define* it as

$$H^b(\rho) = H_I^{d_1}(\rho^I) \frac{\delta \rho_{d_1}^I}{\delta \rho_b} + H_{II}^{d_2}(\rho^{II}) \frac{\delta \rho_{d_2}^{II}}{\delta \rho_b} \tag{16}$$

$$= H_I^{b_1}(\rho^I) \omega^{b_2} + \omega^{b_1} H_{II}^{b_2}(\rho^{II}), \tag{17}$$

with  $b = b_1 b_2$ . (16) is a nonlinear functional generalization of the well known linear recipe  $H_{1+2} = H_1 \otimes \mathbf{1} + \mathbf{1} \otimes H_2$  and reduces to the formulas arising from the triple-bracket formalism if Hamiltonian functions exist. The whole 2-particle equation can be written as follows

$$i\hbar \partial_t \rho_f = \Omega_{abc}^{(2)} \frac{\delta \rho_f}{\delta \rho_a} \left( H_I^{d_1}(\rho^I) \frac{\delta \rho_{d_1}^I}{\delta \rho_b} + H_{II}^{d_2}(\rho^{II}) \frac{\delta \rho_{d_2}^{II}}{\delta \rho_b} \right) \rho^c \tag{18}$$

where  $\rho_f = \rho_{f_1 f_2}$  is the 2-particle density matrix.

Now comes the important general result. Let us perform a partial trace i.e. contract both sides of (18) with  $\omega^{f_2}$ . Since  $\rho_{f_1}^I = \omega^{f_2} \rho_{f_1 f_2} = I_{f_2} \rho_{f_1}^{f_2}$  and  $\omega^{f_2} = \delta(f_2 - f'_2)$  is  $t$ - and  $\rho$ -independent

$$\begin{aligned}
i\hbar \partial_t \rho_{f_1}^I &= \Omega_{abc}^{(2)} \frac{\delta \rho_{f_1}^I}{\delta \rho_a} \left( H_I^{d_1}(\rho^I) \frac{\delta \rho_{d_1}^I}{\delta \rho_b} + H_{II}^{d_2}(\rho^{II}) \frac{\delta \rho_{d_2}^{II}}{\delta \rho_b} \right) \rho^c \\
&= \Omega_{abc}^{(2)} \frac{\delta \rho_{f_1}^I}{\delta \rho_a} H_I^{d_1}(\rho^I) \frac{\delta \rho_{d_1}^I}{\delta \rho_b} \rho^c
\end{aligned} \tag{19}$$

$$= \Omega_{abc}^{(2)} \frac{\delta \rho_{f_1}^I}{\delta \rho_a} H_I^{d_1}(\rho^I) \frac{\delta \rho_{d_1}^I}{\delta \rho_b} \rho^c \tag{20}$$

$$= \Omega_{f_1 d_1 c_1}^{(1)} H_I^{d_1}(\rho^I) \rho^{I c_1}. \tag{21}$$

where the transition from (19) to (20) is a consequence of

$$\Omega_{abc}^{(N)} \frac{\delta \rho_d^I}{\delta \rho_a} \frac{\delta \rho_e^{II}}{\delta \rho_b} = 0 \tag{22}$$

holding for all  $N$ -particle structure constants and reduced density matrices of non-overlapping subsystems (for the proof of (22) see Lemma 1 in [14]).

Let me summarize what has happened until now: Using a correct extension of a 1-particle nonlinear Hamiltonian operator and the general property of triple brackets we have reduced a 2-particle equation for a 2-particle density matrix to a 1-particle equation which involves *only* the quantities which are intrinsic to this subsystem. All elements depending on the other subsystem have simply vanished. We have obtained this by performing only one operation — the partial trace over the “external” subsystem.

Therefore the reduced density matrix in  $I$  does not depend on details of interaction in the separated system  $II$  and the dynamics is strongly separable. Actually, we have simultaneously obtained more. Indeed, we do not assume that we take the partial trace at “ $t = 0$ ”. Therefore we can trace out the external system at any time and the reduced dynamics is indistinguishable from a dynamics defined entirely in terms of the subsystem and starting from the initial condition  $\rho_1(0) = \rho^I(0) = \text{Tr}_2 \rho(0)$  where  $\rho(0)$  is the initial condition for the large system. It proves that the dynamics so constructed is *completely separable*.

The result we have obtained is completely general and works for all Schrödinger equations whose nonlinear Hamiltonian operator kernels allow to write them in terms of 1-particle density matrices. Putting it differently, the construction works correctly if a given Schrödinger equation allows for an extension to an almost-Lie-Poisson Liouville-von Neumann equation. The example of the Haag-Banner equation served only as a means of focusing our attention and making the discussion less abstract.

So how does the 2-particle equation look explicitly? Beginning again with the general formula we can obtain immediately its model independent form just by performing abstract operations on the composite indices. We get

$$\begin{aligned}
i\hbar\partial_t\rho_a &= \Omega_{abc}^{(2)}\left(H_I^{b_1}(\rho^I)\omega^{b_2} + \omega^{b_1}H_{II}^{b_2}(\rho^{II})\right)\rho^c \\
&= \Omega_{a_1b_1c_1}^{(1)}H_I^{b_1}(\rho^I)\rho^{c_1}_{a_2} + \Omega_{a_2b_2c_2}^{(1)}H_{II}^{b_2}(\rho^{II})\rho_{a_1}^{c_2}.
\end{aligned}$$

Returning for the sake of completeness to the Haag-Bannier case we can write it as

$$\begin{aligned}
&i\hbar\partial_t\rho(a_1, a_2, a'_1, a'_2) \\
&= \left[ \left( -\frac{\hbar^2}{2m}(\Delta_{a_1} + \Delta_{a_2} - \Delta_{a'_1} - \Delta_{a'_2}) + V_1(a_1) + V_2(a_2) - V_1(a'_1) - V_2(a'_2) \right. \right. \\
&\quad + \vec{A}_1(a_1) \frac{\int dy_1 \delta(a_1 - y_1) \vec{\nabla}_{a_1} [\int dz \rho(a_1, z, y_1, z) - \int dz \rho(y_1, z, a_1, z)]}{2i \int dz \rho(a_1, z, a_1, z)} \\
&\quad + \vec{A}_2(a_2) \frac{\int dy_2 \delta(a_2 - y_2) \vec{\nabla}_{a_2} [\int dz \rho(z, a_2, z, y_2) - \int dz \rho(z, y_2, z, a_2)]}{2i \int dz \rho(z, a_2, z, a_2)} \\
&\quad - \vec{A}_1(a'_1) \frac{\int dy_1 \delta(a'_1 - y_1) \vec{\nabla}_{a'_1} [\int dz \rho(a'_1, z, y_1, z) - \int dz \rho(y_1, z, a'_1, z)]}{2i \int dz \rho(a'_1, z, a'_1, z)} \\
&\quad \left. \left. - \vec{A}_2(a'_2) \frac{\int dy_2 \delta(a'_2 - y_2) \vec{\nabla}_{a'_2} [\int dz \rho(z, a'_2, z, y_2) - \int dz \rho(z, y_2, z, a'_2)]}{2i \int dz \rho(z, a'_2, z, a'_2)} \right] \right] \rho(a_1, a_2, a'_1, a'_2). \tag{23}
\end{aligned}$$

Its *pure state* 2-particle counterpart is

$$\begin{aligned}
i\hbar\partial_t\Psi(a_1, a_2) &= \left[ -\frac{\hbar^2}{2m}(\Delta_{a_1} + \Delta_{a_2}) + V_1(a_1) + V_2(a_2) \right. \\
&\quad + \vec{A}_1(a_1) \frac{\int dz [\overline{\Psi(a_1, z)} \vec{\nabla}_{a_1} \Psi(a_1, z) - \Psi(a_1, z) \vec{\nabla}_{a_1} \overline{\Psi(a_1, z)}]}{2i \int dz |\Psi(a_1, z)|^2} \\
&\quad \left. + \vec{A}_2(a_2) \frac{\int dz [\overline{\Psi(z, a_2)} \nabla_{a_2} \Psi(z, a_2) - \Psi(z, a_2) \nabla_{a_2} \overline{\Psi(z, a_2)}]}{2i \int dz |\Psi(z, a_2)|^2} \right] \Psi(a_1, a_2). \tag{24}
\end{aligned}$$

This equation has several interesting features. The Hamiltonian operator obviously reduces to the sum of ordinary 1-particle terms, involving no integrals, if  $\Psi$  is a product state. What makes it unusual is the presence of the integrals. Typically it is said that such equations should not be taken into account because they are *nonlocal*. They indeed appear nonlocal but a closer look shows that it is in fact just the opposite: The currents and probability densities are the local 1-particle ones. Therefore it is the lack of *appropriate* integrals that makes typical 2-particle equations nonlocal. The equation considered in [16] involved no such integrals and this led to difficulties. An extension of the above results from 2 to  $N$  particles is immediate so explicit formulas will not be given.

#### IV. FURTHER EXAMPLES

Let me now list some of nonlinear Hamiltonian operators that have been considered in literature and which admit the completely separable extension to an arbitrary number of particles in arbitrary entangled and mixed states. The general rule is that an extension to  $N$  particles will be given in one of these forms but instead of  $\rho(x, x')$  a reduced 1-particle density matrix should be placed. If the  $N$ -particle state is pure then the reduced density matrix is a functional of the pure state which involves  $N - 1$  integrations. This matrix is subsequently put into a suitable place in the Schrödinger equation. Similarly it can be put into a Liouville-von Neumann equation if the state is more general.

a) “*Nonlinear Schrödinger*”

$$|\psi(x)|^2 \rightarrow \rho(x, x)$$

b) *Białynicki-Birula-Mycielski*

$$\ln(|\psi(x)|^2) \rightarrow \ln \rho(x, x)$$

Obviously in the same way one can treat any equation with nonlinearities given by some function  $F(|\psi(x)|)$  [30].

c) *Doebner–Goldin*

$$\begin{aligned}
R_1 &: \frac{1}{2i} \frac{\bar{\psi}(x)\Delta_x\psi(x) - \psi(x)\Delta_x\bar{\psi}(x)}{|\psi(x)|^2} \\
&\rightarrow \frac{1}{2i} \frac{\int dz\delta(x-z)\Delta_x[\rho(x,z) - \rho(z,x)]}{\rho(x,x)}, \\
R_2 &: \frac{\Delta_x|\psi(x)|^2}{|\psi(x)|^2} \rightarrow \frac{\Delta_x\rho(x,x)}{\rho(x,x)} \\
R_3 &: \frac{1}{(2i)^2} \frac{[\bar{\psi}(x)\vec{\nabla}_x\psi(x) - \psi(x)\vec{\nabla}_x\bar{\psi}(x)]^2}{|\psi(x)|^4} \\
&\rightarrow \frac{1}{(2i)^2} \frac{(\int dz\delta(x-z)\vec{\nabla}_x[\rho(x,z) - \rho(z,x)])^2}{\rho(x,x)^2} \\
R_4 &: \frac{1}{2i} \frac{[\bar{\psi}(x)\vec{\nabla}_x\psi(x) - \psi(x)\vec{\nabla}_x\bar{\psi}(x)] \cdot \vec{\nabla}_x|\psi(x)|^2}{|\psi(x)|^4} \\
&\rightarrow \frac{1}{2i} \frac{\int dz\delta(x-z)\vec{\nabla}_x[\rho(x,z) - \rho(z,x)] \cdot \vec{\nabla}_x\rho(x,x)}{\rho(x,x)^2}, \\
R_5 &: \frac{[\vec{\nabla}_x|\psi(x)|^2]^2}{|\psi(x)|^4} \rightarrow \frac{[\vec{\nabla}_x\rho(x,x)]^2}{\rho(x,x)^2}
\end{aligned}$$

d) *Twarock* on  $S^1$  [22]

$$\begin{aligned}
&\frac{\psi(x)''\overline{\psi(x)'} - \overline{\psi(x)''}\psi(x)'}{\psi(x)\overline{\psi(x)'} - \overline{\psi(x)}\psi(x)'} \\
&\rightarrow \frac{[\int dy\delta(x-y)\partial_x^2\rho(x,y)][\int dz\delta(x-z)\partial_x^2\rho(z,x)] - c.c.}{\rho(x,x)\int dy\delta(x-y)\partial_x\rho(x,y) - c.c.}
\end{aligned}$$

e)  $(n, n)$ -homogeneous nonlinearities. Denote by  $D$  a differential operator involving arbitrary mixed partial derivatives up to order  $k$ . Consider a real function  $F(\psi) = F(D\psi(x))$ ,  $(n, n)$ -homogeneous i.e. satisfying  $F(\lambda\psi) = \lambda^n \bar{\lambda}^n F(\psi)$ . We first write

$$F(D\psi(x)) = \frac{F(\overline{\psi(x)}D\psi(x))}{|\psi(x)|^{2n}}$$

and then apply the tricks used for the Haag–Bannier, Doebner–Goldin and Twarock terms. Obviously any reasonable function of such  $(n, n)$ -homogeneous expressions with different  $n$ 's is acceptable as well.

## V. EXTENSION IN STAGES

The formalism proposed above allows to extend a 1-particle dynamics to  $N$  particles. We will now show that it allows for a more general kind of extension: From 1 *system* to  $N$  systems. The procedure is self-consistent in the sense that one can first produce several composite systems from the 1-particle ones, and then combine them into a single overall composite system. Alternatively, one can produce the final system without the intermediate stages, directly by extension from single particles.

Consider the  $N$ -particle extension

$$\begin{aligned}
H_A^b(\rho) &= H_A^{b_1 \dots b_N}(\rho) \\
&= H_{(1)}^{d_1}(\rho^{(1)}) \frac{\delta\rho_{d_1}^{(1)}}{\delta\rho_{b_1 \dots b_N}} + H_{(2)}^{d_2}(\rho^{(2)}) \frac{\delta\rho_{d_2}^{(2)}}{\delta\rho_{b_1 \dots b_N}} + \dots + H_{(N)}^{d_N}(\rho^{(N)}) \frac{\delta\rho_{d_N}^{(N)}}{\delta\rho_{b_1 \dots b_N}} \\
&= H_{(1)}^{b_1}(\rho^{(1)})\omega^{b_2} \dots \omega^{b_N} + \omega^{b_1} H_{(2)}^{b_2}(\rho^{(2)})\omega^{b_3} \dots \omega^{b_N} + \dots + \omega^{b_1} \dots \omega^{b_{N-1}} H_{(N)}^{b_N}(\rho^{(N)}).
\end{aligned} \tag{25}$$

$$\tag{26}$$

The Hamiltonian operator (kernel)  $H_A^b(\rho)$  describes a composite system, labelled  $A$ , which consists of  $N$  particles that do not interact with one another. Consider now another system, labelled  $B$ , consisting of  $M$  particles. Its Hamiltonian operator is

$$H_B^b(\rho) = H_B^{b_{N+1} \dots b_{N+M}}(\rho) \\ = H_{(N+1)}^{d_{N+1}}(\rho^{(N+1)}) \frac{\delta \rho_{d_{N+1}}^{(N+1)}}{\delta \rho_{b_{N+1} \dots b_{N+M}}} + H_{(N+2)}^{d_{N+2}}(\rho^{(N+2)}) \frac{\delta \rho_{d_{N+2}}^{(N+2)}}{\delta \rho_{b_{N+1} \dots b_{N+M}}} + \dots + H_{(N+M)}^{d_{N+M}}(\rho^{(N+M)}) \frac{\delta \rho_{d_{N+M}}^{(N+M)}}{\delta \rho_{b_{N+1} \dots b_{N+M}}} \quad (27)$$

$$= H_{(N+1)}^{b_{N+1}}(\rho^{(N+1)}) \omega^{b_{N+2}} \dots \omega^{b_{N+M}} + \omega^{b_{N+1}} H_{(N+2)}^{b_{N+2}}(\rho^{(N+2)}) \omega^{b_{N+3}} \dots \omega^{b_{N+M}} \\ + \dots + \omega^{b_{N+1}} \dots \omega^{b_{N+M-1}} H_{(N+M)}^{b_{N+M}}(\rho^{(N+M)}) \quad (28)$$

The  $(N+M)$ -particle extension can be obtained in two stages

$$H_{A+B}^b(\rho) = H_{A+B}^{b_1 \dots b_{N+M}}(\rho) \\ = H_A^{d_1 \dots d_N}(\rho^A) \frac{\delta \rho_{d_1 \dots d_N}^A}{\delta \rho_{b_1 \dots b_{N+M}}} + H_B^{d_{N+1} \dots d_{N+M}}(\rho^B) \frac{\delta \rho_{d_{N+1} \dots d_{N+M}}^B}{\delta \rho_{b_1 \dots b_{N+M}}} \\ = H_A^{b_1 \dots b_N}(\rho^A) \omega^{b_{N+1}} \dots \omega^{b_{N+M}} + \omega^{b_1} \dots \omega^{b_N} H_B^{b_{N+1} \dots b_{N+M}}(\rho^B) \\ = \left( H_{(1)}^{b_1}(\rho^{(1)}) \omega^{b_2} \dots \omega^{b_N} + \dots + \omega^{b_1} \dots \omega^{b_{N-1}} H_{(N)}^{b_N}(\rho^{(N)}) \right) \omega^{b_{N+1}} \dots \omega^{b_{N+M}} \\ + \omega^{b_1} \dots \omega^{b_N} \left( H_{(N+1)}^{b_{N+1}}(\rho^{(N+1)}) \omega^{b_{N+2}} \dots \omega^{b_{N+M}} + \dots + \omega^{b_{N+1}} \dots \omega^{b_{N+M-1}} H_{(N+M)}^{b_{N+M}}(\rho^{(N+M)}) \right) \quad (29)$$

which could be derived also directly from the  $(N+M)$ -particle extension of the 1-particle dynamics.

## VI. IS NONLINEAR QUANTUM MECHANICS NONLOCAL?

N. Gisin in his nowadays classic paper [8] argued that *any* nonlinear and non-stochastic dynamics necessarily leads to faster-than-light communication via EPR-type correlations. The argument is based (implicitly) on the additional assumption that a reduced density matrix (of, say, Alice) evolves by means of an independent dynamics of its pure-state components. (Let us note that this was essentially the main question discussed by Haag and Bannier [3] in the context of the nonlinear, convex scheme proposed by Mielnik [32].) This viewpoint is suggested by the fact that what one starts with is a nonlinear (Schrödinger-type) dynamics of pure states. But in the EPR case the pure state is a 2-particle one and one has to start with a 2-particle nonlinear Schrödinger equation. Assume this equation is given (e.g. our Haag-Bannier equation) and its 2-particle pure-state solution  $|\psi\rangle$  has been found. To have the Gisin effect one has to find at least one observable of the form

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} \otimes 1 | \psi \rangle = \text{Tr}_A \rho_A \hat{A} \quad (30)$$

describing the Alice subsystem, which depends on a parameter controlled by Bob. Here  $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$  is the reduced density matrix of the Alice subsystem. Assume that the only actions Bob can undertake are reducible to the modifications of the parameters of the Hamiltonian corresponding to “his” particle. (In the case contemplated by Gisin Bob would rotate his Stern-Gerlach device; this is equivalent to modifying the magnetic term in the corresponding interaction Hamiltonian.) (30) shows that Bob can influence  $\langle \hat{A} \rangle$  if and only if the reduced density matrix  $\rho_A$  depends on the parameter he controls. This is true for the Weinberg theory [10] but will *never* happen for *any* equation we have discussed. It follows that the Gisin effect applies to a limited class of theories.

The additional assumption that leads to the Gisin phenomenon is the following: Each time Bob makes a measurement, he chooses an initial condition for the reduced density matrix of Alice. This explicitly involves the *ordinary* reduction of the wave packet postulate. One can invent other versions of the postulate, for example, a composition of a nonlinear map  $N$  with a projector  $P$ :  $N^{-1} \circ P \circ N$ , as proposed by Lücke in his analysis of nonlinear gauge transformations [31].

Alternatively, as in the Weinberg theory, one can get the Gisin effect *without* the explicit use of the projection postulate: The nonlinear generator of evolution must be basis dependent. If this is the case, the 2-particle solution is parametrized by the choice of the basis in the Hilbert space. Assuming that Bob can change this basis we allow him to change globally the form of the solution. But the rotation of the Stern-Gerlach device (modification of the interaction term) is not yet a change of basis in this sense. In the Weinberg theory the generator was interpreted as

an average energy. A change of basis was equivalent to a different way of measuring the average. For this reason it was justified to say [12] that the effect was implied by the projection postulate although explicit calculations were not referring to any projections [10] (similarly to the linear case, the projection is not included in a non-stochastic Schrödinger dynamics).

We can conclude that the Gisin-type reasoning has to be regarded as “unphysical”, exactly in the same sense as the Einstein-Podolsky-Rosen argument was unphysical according to Bohr. The version of nonlinear theory we have discussed seems to be free of physical inconsistencies and is not more nonlocal than the linear one.

The idea of using the Haag-Bannier equation as a laboratory for testing the concepts of separability was suggested to me by G. A. Goldin who also informed me about the results of his joint work with G. Svetlichny. I am indebted to G. A. Goldin, H.-D. Doebner, W. Lücke, P. Nattermann, and J. Hennig for valuable suggestions and critical comments, and to W. Puszkarz who explained to me how to include into the framework the equations of Kostin [33] and Staruszkiewicz [34,35]. The work is a part of the Polish-Flemish project 007 and was done during my stay at the Arnold Sommerfeld Institute in Clausthal. I gratefully acknowledge a support from DAAD.

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$$\begin{aligned} & F\left([\int dz |\psi(x_1)\phi(z)|^2]^{1/2}\right) + F\left([\int dz |\psi(z)\phi(x_2)|^2]^{1/2}\right) \\ &= F(|\psi(x_1)|) + F(|\phi(x_2)|) \end{aligned}$$

as expected. In [2] it was assumed that the two particle nonlinearity is  $F(|\Psi(x_1, x_2)|)$ .

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